Structural Design Sensitivity Analysis with Generalized Global Stiffness and Mass Matrices

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First- and second-order derivatives of measures of structural response with respect to design variables are calculated using the generalized global stiffness and mass matrix formulation of structural equations. The adjoint variable method is extended for first-order design sensitivity calculation and a mixed direct differentiation-adjoint variable method is employed to calculate second derivatives of static structural response measures. The variational or virtual work form of the structural equations is shown to be well suited for design sensitivity analysis with generalized global stiffness and mass matrices, avoiding the requirement for explicit reduction of the matrices. The method is illustrated using a simple structure with a multipoint constraint.

I. Introduction

DESIGN sensitivity analysis of structural response to an applied load has been extensively developed in the literature; e.g., Refs. 1-3 and literature cited therein. Virtually all design sensitivity analysis for structures described by matrix finite element equations has been carried out using reduced global stiffness and mass matrices that are obtained by eliminating displacement coordinates that are specified by boundary conditions, so they are positive definite and have inverses. Most derivations of design sensitivity make essential use of the inverse of the stiffness matrix. This approach presents a difficulty in implementing design sensitivity calculations with computer codes in which the reduced global matrices are not constructed. Numerical reduction techniques that account for boundary conditions, particularly multipoint boundary conditions, generate numerical solutions of the structural equations without explicitly constructing the reduced system stiffness and mass matrices.

In contrast to design sensitivity derivations that use the inverse of the system stiffness matrix, distributed parameter structural design sensitivity analysis[4] is carred out using a variational formulation that avoids introduction of reduced global stiffness and mass matrices. The purpose of this paper is to present a variational formulation of matrix structural design sensitivity analysis explicitly in terms of the generalized global stiffness and mass matrices.

II. Variational Formulation with Generalized Global Stiffness and Mass Matrices

Denoting the vector of global generalized coordinates as $z_g = [z_1,\ldots,z_n]^T$, one may write the total potential energy of a structure in the form

$$\mathrm{PE} = \tfrac{1}{2} z_g^T K_g(b) z_g - z_g^T f_g(b) \tag{1}$$

where $K_g(b)$ is the positive semidefinite generalized global stiffness matrix that defines the strain energy quadratic form and $f_g(b)$ is the vector of global generalized nodal forces. Note that both the stiffness matrix and load vector may depend on design, characterized by the design parameter vector $b = [b_1,\ldots,b_k]^T$. If one defines the space Z of kinematically admissible displacements (i.e., the collection of all generalized displacement vectors satisfying homogeneous boundary conditions), then on the space of kinematically admissible displacements the strain energy quadratic form is positive definite; that is,

$$z_g^T K_g(b) z_g > 0, \qquad z_g \neq 0, \quad z_g \in Z \tag{2}$$

Note that since Z is a proper subspace of R^n, $K_g(b)$ is not positive definite and hence is singular.

The theorem of minimum total potential energy may be used to define necessary and sufficient conditions for z_g to be the solution to the static structural problem. Since the total potential energy function of Eq. (1) is strictly convex on the space Z of kinematically admissible displacements, there exists a unique minimum of PE, given by $z_g \in Z$. Letting $\bar{z}_g \in Z$ be a kinematically admissible virtual displacement, one may write neighboring states of the structural system as $z_g^* = z_g + \epsilon \bar{z}_g$, where ϵ is a small real number. Substituting z_g^* into Eq. (1), taking the derivative with respect to ϵ, and setting the result equal to 0 at $\epsilon = 0$ (the necessary condition for a minimum), one has the variational equation

$$\bar{z}_g^T K_g(b) z_g = \bar{z}_g^T f_g(b), \qquad \text{for all } \bar{z}_g \in Z \tag{3}$$

Note that this equation must hold for all kinematically admissible virtual displacements $\bar{z}_g$. By strict convexity of the total potential energy on Z, Eq. (3) is guaranteed to have a unique solution in Z. From a physical point of view, one may interpret Eq. (3) as the equation of virtual work for the structure, the left side being the virtual work of the internal forces and the right side being the virtual work of the externally applied forces.

The variational form of Lagrange's equations of motion may similarly be employed to derive the variational eigenvalue problem

$$\bar{y}_g^T K_g(b) y_g = \zeta \bar{y}_g^T M_g(b) y_g, \qquad \text{for all } \bar{y}_g \in Z \tag{4}$$

for natural vibration of the structure, where $M_g(b)$ is the generalized mass matrix, $\zeta = \omega^2$, and the eigenvector y_g is normalized by the condition

$$y_g^T M_g(b) y_g = 1 \tag{5}$$

Since the generalized mass matrix M_g is positive definite and the strain energy quadratic form is positive definite on Z,

Received May 1, 1983; revision received Oct. 27, 1983.

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some attractive mathematical properties are associated with the variational equation given by Eq. (4).[3]

III. Reduced Stiffness Matrix in Design Sensitivity Analysis

Before carrying out design sensitivity analysis with the generalized global stiffness and mass matrices, it is instructive to briefly review the derivation for static response in terms of a reduced system stiffness matrix. If one explicitly eliminates generalized coordinates that are specified by boundary conditions, a reduced system stiffness matrix $K(b)$ of dimension $m<n$ is obtained. The matrix structural equation is then

$$K(b)z=f(b) \tag{6}$$

where $K(b)$ is positive definite, hence nonsingular. By the implicit function theorem of calculus, if elements of the matrix $K(b)$ and vector $f(b)$ are twice continuously differentiable with respect to b, then the state z is twice continuously differentiable with respect to b. One may thus take the derivative of both sides of Eq. (6) to obtain

$$K(b)\frac{\mathrm{d}z}{\mathrm{d}b}=\frac{\mathrm{d}f}{\mathrm{d}b}-\frac{\partial}{\partial b}(K(b)\tilde{z}) \tag{7}$$

where matrix calculus notation $\mathrm{d}z/\mathrm{d}b=[\partial z_i/\partial b_j]$ is used and a tilde denotes a variable that is treated as constant in the partial differentiation.

Consider a structural performance measure $\psi(z,b)$, which may represent weight of the structure, stress in a structural component, displacement of a structural component, or some other measure of structural response to load. Note that since the solution z of Eq. (6) depends on design, ψ is dependent on design through both of its arguments. One may thus use the chain rule of differential calculus to write

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}b}\psi(z,b)&=\frac{\partial\psi}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}b}+\frac{\partial\psi}{\partial b}\\&=\frac{\partial\psi}{\partial z}K^{-1}(b)\left[\frac{\mathrm{d}f}{\mathrm{d}b}-\frac{\partial}{\partial b}(K(b)\tilde{z})\right]+\frac{\partial\psi}{\partial b}\\&=\lambda^T\frac{\mathrm{d}f}{\mathrm{d}b}-\frac{\partial}{\partial b}(\tilde{\lambda}^TK(b)\tilde{z})+\frac{\partial\psi}{\partial b}\end{aligned} \tag{8}$$

where Eq. (7) has been solved for $\mathrm{d}z/\mathrm{d}b$ in the second line and an adjoint variable λ has been introduced in the third line, defined as the solution of

$$K(b)\lambda=\frac{\partial\psi^T}{\partial z} \tag{9}$$

This well-known result provides a computable formula for the design derivative of a structural response measure, requiring solution of Eq. (6) for structural displacement z and Eq. (9) for the adjoint variable λ. Note, however, that the inverse of the reduced system stiffness matrix was used in derivation of Eqs. (8) and (9). As noted earlier, if the reduced system stiffness matrix $K(b)$ is not explicitly evaluated in a finite element analysis, then this result is of no value.

IV. Reduction of Variational Formulations to Matrix Form

Equations (3) and (4) can be used to generate matrix equations for constructing numerical solutions. Let $\phi^i\in Z\subset R^n$, $i=1,\ldots,m$, $m<n$, be a basis of the vector space Z of kinematically admissible displacements. Then the solution of Eq. (3) may be written as

$$z_g=\sum_{j=1}^{m}c_j\phi^j=\Phi c \tag{10}$$

where $\Phi=[\phi^1,\ldots,\phi^m]$ and the coefficients c_j are uniquely determined. Substituting this representation for z_g into Eq. (3) and evaluating Eq. (3) at $\bar{z}_g=\phi^i$, $i=1,\ldots,m$, one has the equations

$$\sum_{j=1}^{m}(\phi^{i^T}K_g\phi^j)c_i=\phi^{i^T}f_g,\qquad i=1,\ldots,m \tag{11}$$

Defining the matrices

$$\begin{aligned}\check{K}&=[\phi^{i^T}K_g\phi^j]_{m\times m}=\Phi^TK_g\Phi\\\check{f}&=[\phi^{i^T}f_g]_{m\times 1}=\Phi^Tf_g\\c&=[c_i]_{m\times 1}\end{aligned} \tag{12}$$

Eq. (11) may be written in matrix form as

$$\check{K}(b)c=\check{f}(b) \tag{13}$$

It is easy to see that this equation has a unique solution c, since $\check{K}$ is positive definite (due to the assumption of positive definiteness of the strain energy quadratic form on Z). It is also clear that the matrices $\check{K}$ and $\check{f}$ depend on the choice of basis of the space Z of kinematically admissible displacements. Different choices of bases yield different matrices, but the resulting solution for z_g, given by Eq. (10), is unique.

Just as in the case of equilibrium of the structure, the variational equation of Eq. (4) can be reduced to a matrix equation, using a basis for the space Z of kinematically admissible displacements. This yields a generalized eigenvalue problem

$$\check{K}(b)c=\zeta\check{M}(b)c \tag{14}$$

where components of the vector c are coefficients of Eq. (10). By expanding the eigenvector y_g in terms of the basis ϕ^i, one obtains

$$\check{M}=[\phi^{i^T}M_g\phi^j]_{m\times m}=\Phi^TM_g\Phi \tag{15}$$

As will normally be the case, the matrix M_g is positive definite, so the matrices $\check{M}$ and $\check{K}$ are positive definite, yielding important theoretical and computational properties.[3]

Before carrying out design sensitivity analysis with the variational formulation, one should show that the generalized global displacement vector z_g, the eigenvector y_g, and the eigenvalue ζ are differentiable with respect to design.

Consider an explicit form for the vector space Z of kinematically admissible displacements, given by

$$Z=\{z_g\in R^n : Gz_g=0\} \tag{16}$$

where G is an $(n-m)\times n$ matrix that does not depend on design. With a fixed basis ϕ^i, $i=1,\ldots,m$, of Z that is independent of design, one may represent the solution z_g in the form of Eq. (10), where the vector c of coefficients is determined by Eq. (13). Note that the dependence of $\check{K}$ and $\check{f}$ on design b is explicitly defined in terms of $K_g(b)$ and $f_g(b)$ in Eq. (12). Therefore, $\check{K}(b)$ and $\check{f}(b)$ are differentiable with respect to design and $\check{K}(b)$ is nonsingular in a neighborhood of the nominal design. The derivatives of the vector c with respect to design thus exist, as shown by the method in Sec. III. Once $\mathrm{d}c/\mathrm{d}b$ is determined, one may use Eq. (10) to obtain

$$\frac{dz_g}{db} = \Phi \frac{dc}{db} \tag{17}$$

since Φ does not depend on b. Thus, the question of differentiability of z_g is resolved.

For differentiability of the generalized global eigenvector y_g and eigenvalue ζ one can argue exactly, as in the case of the generalized global displacement vector z_g, using the results of Sec. III and Eq. (14).

V. Design Sensitivity Analysis with the Variational Formulation

In case the generalized global formulation of Sec. II is employed, the measure of structural response $\psi(z_g,b)$ will be written in terms of the generalized global displacement vector z_g. Its total derivative with respect to design is thus

$$\frac{d}{db}\psi(z_g,b) = \frac{\partial \psi}{\partial z_g}\frac{dz_g}{db} + \frac{\partial \psi}{\partial b} \tag{18}$$

In order to obtain information about dz_g/db, the total derivative of both sides of Eq. (3) may be calculated, where the virtual displacement $\bar{z}_g$ does not depend on design, to obtain

$$\bar{z}_g^T K_g(b)\frac{dz_g}{db} = \bar{z}_g^T \frac{df_g(b)}{db} - \frac{\partial}{\partial b}(\bar{z}_g^T K_g(b)\tilde{z}_g), \quad \text{for all } \bar{z}_g \in Z \tag{19}$$

To rewrite the first term on the right of Eq. (19), one may view it as a row vector of virtual work expressions, with $\partial z_g/\partial b_i$, $i=1,...,k$, playing the role of virtual displacements, and define the adjoint equation

$$\lambda_g^T K_g(b)\bar{\lambda}_g = \frac{\partial \psi}{\partial z_g}\bar{\lambda}_g, \qquad \text{for all } \bar{\lambda}_g \in Z \tag{20}$$

where $dz_g/db = [\partial z_g/\partial b_1,...,\partial z_g/\partial b_k]$ and use has been made of the fact that the generalized global stiffness matrix $K_g(b)$ is symmetric. Equation (20) may be solved for λ_g, using the same reduction technique that is used to obtain a solution of Eq. (3) for displacement. Note that $\lambda_g \in Z$ and may be viewed as a displacement due to a load $(\partial\psi/\partial z_g)^T$. To make use of the identities of Eqs. (19) and (20), Eq. (20) may be evaluated at $\bar{\lambda}_g = \partial z_g/\partial b_i$, $i=1,...,k$, and Eq. (19) at $\bar{z}_g = \lambda_g$, to obtain

$$\frac{\partial \psi}{\partial z_g}\frac{dz_g}{db} = \lambda_g^T K_g(b)\frac{dz_g}{db}$$

$$= \lambda_g^T \frac{df_g(b)}{db} - \frac{\partial}{\partial b}(\tilde{\lambda}_g^T K_g(b)\tilde{z}_g) \tag{21}$$

Note that with this manipulation the first term on the right of Eq. (18) may be written explicitly in terms of derivatives with respect to design and the adjoint variable λ_g that is obtained as the solution of Eq. (20). Substituting from Eq. (21) into Eq. (18),

$$\frac{d\psi}{db} = \lambda^T \frac{df_g(b)}{db} - \frac{\partial}{\partial b}(\tilde{\lambda}_g^T K_g(b)\tilde{z}_g) + \frac{\partial \psi}{\partial b} \tag{22}$$

Note that the final design sensitivity formula in Eq. (22) is identical in form to that obtained in Eq. (8) for the reduced formulation. However, calculations in Eq. (22) may be carried out directly in terms of derivatives of the generalized applied load and generalized global stiffness matrix. As noted previously, this is essential in finite element formulations in which the reduced load and global stiffness matrices are not constructed, hence making the theoretical calculation in Eq. (8) meaningless.

For design sensitivity analysis of a simple eigenvalue, the total derivative of both sides of Eq. (4), with $\bar{y}_g$ independent of design, yields

$$\bar{y}_g^T K_g(b)\frac{dy_g}{db} + \frac{\partial}{\partial b}(\bar{y}_g^T K_g(b)\tilde{y}_g) = \frac{d\zeta}{db}\bar{y}_g^T M_g(b)y_g$$

$$+ \zeta \bar{y}_g^T M_g(b)\frac{dy_g}{db} + \zeta\frac{\partial}{\partial b}(\bar{y}_g^T M_g(b)\tilde{y}_g), \quad \text{for all } \bar{y}_g \in Z \tag{23}$$

Since Eq. (23) must hold for all $\bar{y}_g \in Z$, one may substitute $\bar{y}_g = y_g$ in Eq. (23) and use Eq. (5) to obtain

$$\frac{d\zeta}{db} = \frac{\partial}{\partial b}(\bar{y}_g^T K_g(b)\tilde{y}_g) - \zeta\frac{\partial}{\partial b}(\bar{y}_g^T M_g(b)\tilde{y}_g)$$

$$+ [K_g(b)y_g - \zeta M_g(b)y_g]^T \frac{dy_g}{db} \tag{24}$$

Note that the last term in Eq. (24) is zero, since y_g is a solution of Eq. (4) and $dy_g/db_i = \Phi(dc/db_i) \in Z$, $i=1,2,...,k$. Thus, Eq. (24) reduces to the desired result

$$\frac{d\zeta}{db} = \frac{\partial}{\partial b}(\bar{y}_g^T K_g(b)\tilde{y}_g) - \zeta\frac{\partial}{\partial b}(\bar{y}_g^T M_g(b)\tilde{y}_g) \tag{25}$$

Note that, as in the static response case, the design sensitivity formula in Eq. (25) is identical in form to that obtained in the literature for the reduced formulation. However, as noted in the static response case, calculations in Eq. (25) can be carried out directly, whereas the reduced stiffness and mass matrices may not be available.

If multiple (repeated) eigenvalues are encountered, the design sensitivity formula of Eq. (25) is not valid. It is shown in Ref. 3 that for an s-times repeated eigenvalue ζ, the directional derivatives[3] of ζ in the direction δb are the eigenvalues of the $s \times s$ matrix

$$\mathfrak{M} = \left[\frac{\partial}{\partial b}(\bar{y}_g^{iT} K_g(b)\tilde{y}_g^j)\delta b - \zeta\frac{\partial}{\partial b}(\bar{y}_g^{iT} M_g(b)\tilde{y}_g^j)\delta b\right]_{s\times s} \tag{26}$$

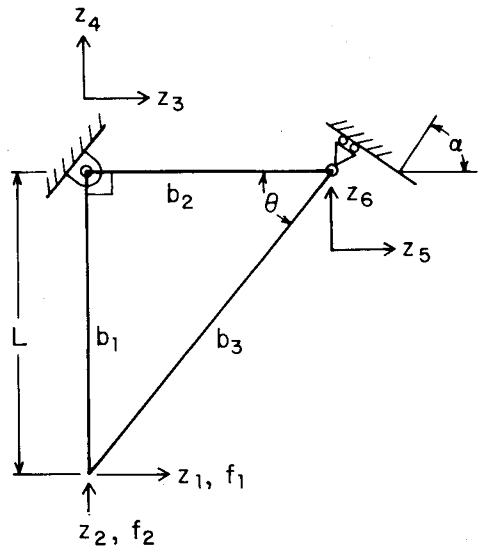

Fig. 1 Three bar truss.

where y_g^i, $i=1,2,\ldots,s$, are $M_g(b)$-orthonormal eigenvectors of Eq. (4); i.e.,

$$y_g^{i^T} M_g(b) y_g^j = \delta_{ij} \tag{27}$$

where $\delta_{ij}=1$ if $i=j$ and $\delta_{ij}=0$ otherwise.

VI. Second-Order Design Sensitivity Analysis

Consider now calculation of second derivatives of $\psi(z_g,b)$ with respect to design. A recent Note[2] and a refinement by Haftka[5] are extended here to provide results in terms of the generalized global stiffness matrix and load vector. Second derivatives of ψ with respect to design may be written in the form

$$\frac{\mathrm{d}^2}{\mathrm{d}b_i\mathrm{d}b_j}\psi(z_g,b)=\frac{\partial\psi}{\partial z_g}\frac{\mathrm{d}^2z_g}{\mathrm{d}b_i\mathrm{d}b_j}+\frac{\partial^2\psi}{\partial z_g\partial b_j}\frac{\mathrm{d}z_g}{\mathrm{d}b_i}+\frac{\mathrm{d}z_g^T}{\mathrm{d}b_j}\frac{\partial^2\psi}{\partial z_g^2}\frac{\mathrm{d}z_g}{\mathrm{d}b_i}+\frac{\partial^2\psi}{\partial b_i\partial z_g}\frac{\mathrm{d}z_g}{\mathrm{d}b_j}+\frac{\partial^2\psi}{\partial b_i\partial b_j} \tag{28}$$

where the total derivative notation on the left is used to emphasize inclusion of design dependence of z_g that appears in the performance measure. In order to treat the first term on the right of Eq. (28), consider the ith component of Eq. (19) and differentiate both sides with respect to b_j to obtain the identity

$$\bar{z}_g^T K_g(b)\frac{\mathrm{d}^2z_g}{\mathrm{d}b_i\mathrm{d}b_j}=\bar{z}_g^T\frac{\partial^2 f_g(b)}{\partial b_i\partial b_j}-\frac{\partial^2}{\partial b_i\partial b_j}\left(\bar{z}_g^T K_g(b)\tilde{z}_g\right)-\frac{\partial}{\partial b_i}\left(\bar{z}_g^T K_g(b)\frac{\mathrm{d}\tilde{z}_g}{\mathrm{d}b_j}\right)-\frac{\partial}{\partial b_j}\left(\bar{z}_g^T K_g(b)\frac{\mathrm{d}\tilde{z}_g}{\mathrm{d}b_i}\right),$$

$$\text{for all } \bar{z}_g\in Z \tag{29}$$

A clever idea introduced by Haftka[5] is to evaluate Eq. (20) at $\bar{\lambda}_g=\mathrm{d}^2z_g/\mathrm{d}b_i\mathrm{d}b_j$ and Eq. (29) at $\bar{z}_g=\lambda_g$, to obtain an expression for the first term on the right of Eq. (28). Making these substitutions into Eq. (28),

$$\frac{\mathrm{d}^2}{\mathrm{d}b_i\mathrm{d}b_j}\psi(z_g,b)=\lambda_g^T\frac{\partial^2 f_g(b)}{\partial b_i\partial b_j}-\frac{\partial^2}{\partial b_i\partial b_j}\left(\tilde{\lambda}_g^T K_g(b)\tilde{z}_g\right)-\frac{\partial}{\partial b_i}\left(\tilde{\lambda}_g^T K_g(b)\frac{\mathrm{d}\tilde{z}_g}{\mathrm{d}b_j}\right)-\frac{\partial}{\partial b_j}\left(\tilde{\lambda}_g^T K_g(b)\frac{\mathrm{d}\tilde{z}_g}{\mathrm{d}b_i}\right)+\frac{\partial^2\psi}{\partial z_g\partial b_j}\frac{\mathrm{d}z_g}{\mathrm{d}b_i}+\frac{\mathrm{d}z_g^T}{\mathrm{d}b_j}\frac{\partial^2\psi}{\partial z_g^2}\frac{\mathrm{d}z_g}{\mathrm{d}b_i}+\frac{\partial^2\psi}{\partial b_i\partial z_g}\frac{\mathrm{d}z_g}{\mathrm{d}b_j}+\frac{\partial^2\psi}{\partial b_i\partial b_j} \tag{30}$$

Instead of using an adjoint variable technique to eliminate $\mathrm{d}z_g/\mathrm{d}b_i$, as was done in Ref. 2, Eq. (19) may be solved k times for $\mathrm{d}z_g/\mathrm{d}b_i$, $i=1,\ldots,k$. To evaluate the second derivatives in Eq. (30), one need only solve Eq. (3) for the state z_g, Eq. (20) for λ_g, and Eq. (19) for the first derivatives of z_g with respect to design. Thus, $k+2$ solutions of the same structural equation, with only different applied loads, yield the complete evaluation of second derivatives in Eq. (30). This is a remarkably small computational penalty for calculating the full set of second derivatives of a general performance measure that depends on state and design.

VII. Example

To illustrate results of previous sections, consider a simple three bar truss with multipoint boundary conditions, as shown in Fig. 1. Design variables for the structure are the cross-sectional areas b_i of the truss members. The generalized global stiffness matrix is

$$K_g(b)=\frac{E}{L}\times\begin{bmatrix} b_3c^2s & b_3cs^2 & 0 & 0 & -b_3c^2s & -b_3cs^2 \\ b_3cs^2 & b_1+b_3s^3 & 0 & -b_1 & -b_3cs^2 & -b_3s^3 \\ 0 & 0 & \frac{b_2s}{c} & 0 & -\frac{b_2s}{c} & 0 \\ 0 & -b_1 & 0 & b_1 & 0 & 0 \\ -b_3c^2s & -b_3cs^2 & -\frac{b_2s}{c} & 0 & \frac{b_2s}{c}+b_3c^2s & b_3cs^2 \\ -b_3cs^2 & -b_3s^3 & 0 & 0 & b_3cs^2 & b_3s^3 \end{bmatrix} \tag{31}$$

and the generalized global lumped mass matrix is

$$M_g(b)=\frac{\rho L}{2}\times\begin{bmatrix} b_1+\sqrt{2}b_3 & & & & & \\ & b_1+\sqrt{2}b_3 & & & 0 & \\ & & b_1+b_2 & & & \\ & 0 & & b_1+b_2 & & \\ & & & & b_2+\sqrt{2}b_3 & \\ & & & & & b_2+\sqrt{2}b_3 \end{bmatrix} \tag{32}$$

where $c=\cos\theta$, $s=\sin\theta$, and $\rho=$ mass density. In this problem, the space Z of kinematically admissible displacements is

$$Z=\{z_g\epsilon R^6\,|\,z_3=z_4=0,\ z_5\cos\alpha+z_6\sin\alpha=0\} \tag{33}$$

and $K_g(b)$ is positive definite on Z.

If $\theta=45$ deg and $\alpha=30$ deg, then with $z=[z_1,z_2,z_5]^T$ the reduced stiffness and mass matrices in this elementary example are

$$K(b)=\frac{E}{2\sqrt{2}L}\begin{bmatrix} b_3 & b_3 & (\sqrt{3}-1)b_3 \\ b_3 & 2\sqrt{2}b_1+b_3 & (\sqrt{3}-1)b_3 \\ (\sqrt{3}-1)b_3 & (\sqrt{3}-1)b_3 & 2\sqrt{2}b_2+(4-2\sqrt{3})b_3 \end{bmatrix} \tag{34}$$

and

$$M(b)=\frac{\rho L}{2}\begin{bmatrix} b_1+\sqrt{2}b_3 & 0 & 0 \\ 0 & b_1+\sqrt{2}b_3 & 0 \\ 0 & 0 & 4(b_2+\sqrt{2}b_3) \end{bmatrix} \tag{35}$$

If $f_1=f_2=1$ and $L=1$, then the solution of the reduced stiffness matrix formulation of Eq. (6) is

$$z=\left[\frac{4-2\sqrt{3}}{Eb_2}+\frac{2\sqrt{2}}{Eb_3},\ 0,\ \frac{1-\sqrt{3}}{Eb_2}\right]^T \tag{36}$$

If $\psi=z_1$, then Eq. (9) is

$$K(b)\lambda=\frac{\partial\psi^T}{\partial z}=[1,\ 0,\ 0]^T \tag{37}$$

with solution

$$\lambda=\left[\frac{1}{Eb_1}+\frac{4-2\sqrt{3}}{Eb_2}+\frac{2\sqrt{2}}{Eb_3},\ -\frac{1}{Eb_1},\ \frac{1-\sqrt{3}}{Eb_2}\right]^T \tag{38}$$

The reduced stiffness matrix design sensitivity formula of Eq. (8) gives, using z and λ from Eqs. (36) and (38),

$$\frac{d\psi}{db}=-\frac{\partial}{\partial b}(\tilde{\lambda}^T K(b)\tilde{z})=\left[0,\ \frac{2\sqrt{3}-4}{Eb_2^2},\ -\frac{2\sqrt{2}}{Eb_3^2}\right] \tag{39}$$

This can be verified by taking the derivative of z_1 in Eq. (36) with respect to design parameter b.

If the generalized global formulation is employed, one must find the solution z_g of Eq. (3), which is

$$z_g=\left[\frac{4-2\sqrt{3}}{Eb_2}+\frac{2\sqrt{2}}{Eb_3},\ 0,\ 0,\ 0,\ \frac{1-\sqrt{3}}{Eb_2},\ \frac{3-\sqrt{3}}{Eb_2}\right]^T \tag{40}$$

For $\psi=z_1$, Eq. (20) is

$$\lambda_g^T K_g(b)\bar{\lambda}_g=[1,\ 0,\ 0,\ 0,\ 0,\ 0]^T\bar{\lambda}_g, \qquad \text{for all } \bar{\lambda}_g\in Z \tag{41}$$

with solution

$$\lambda_g=\left[\frac{1}{Eb_1}+\frac{4-2\sqrt{3}}{Eb_2}+\frac{2\sqrt{2}}{Eb_3},\ -\frac{1}{Eb_1},\ 0,\ 0,\ \frac{1-\sqrt{3}}{Eb_2},\ \frac{3-\sqrt{3}}{Eb_2}\right]^T \tag{42}$$

Then the design sensitivity formula of Eq. (22) gives

$$\frac{d\psi}{db}=-\frac{\partial}{\partial b}(\tilde{\lambda}_g^T K_g(b)\tilde{z}_g)=\left[0,\ \frac{2\sqrt{3}-4}{Eb_2^2},\ -\frac{2\sqrt{2}}{Eb_3^2}\right] \tag{43}$$

which is identical to the result obtained in Eq. (39).

For second-order design sensitivity, solving Eq. (19) for dz_g/db_i, $i=1,2,3$, one has

$$\frac{dz_g}{db_1}=[0,\ 0,\ 0,\ 0,\ 0,\ 0]^T$$

$$\frac{dz_g}{db_2}=\left[\frac{2\sqrt{3}-4}{Eb_2^2},\ 0,\ 0,\ 0,\ \frac{\sqrt{3}-1}{Eb_2^2},\ \frac{\sqrt{3}-3}{Eb_2^2}\right]^T$$

$$\frac{dz_g}{db_3}=\left[-\frac{2\sqrt{2}}{Eb_3^2},\ 0,\ 0,\ 0,\ 0,\ 0\right]^T \tag{44}$$

For $\psi=z_1$, from Eq. (30), one has

$$\frac{d^2\psi}{db_2^2}=-2\frac{\partial}{\partial b_2}\left(\tilde{\lambda}_g^T K_g(b)\frac{d\tilde{z}_g}{db_2}\right)=\frac{8-4\sqrt{3}}{Eb_2^3} \tag{45}$$

and

$$\frac{d^2\psi}{db_3^2}=-2\frac{\partial}{\partial b_3}\left(\tilde{\lambda}_g^T K_g(b)\frac{d\tilde{z}_g}{db_3}\right)=\frac{4\sqrt{2}}{Eb_3^3} \tag{46}$$

The remaining second derivatives are zero. Hence, the Hessian of ψ is a diagonal matrix.

For the eigenvalue problem, assume $E=1$, $\rho=1$, $b_1=b_2=1$, and $b_3=2\sqrt{2}$. Then, the fundamental eigenvalue is $\zeta=0.08038$ and the $M(b)$ – normalized eigenvector is

$$y=[y_1,\ y_2,\ y_5]^T=[-0.3496,\ 0.08451,\ 0.2601]^T \tag{47}$$

The reduced eigenvalue design sensitivity may now be evaluated from Eq. (25) as

$$\frac{d\zeta}{db}=\frac{\partial}{\partial b}(\tilde{y}^T K(b)\tilde{y})-\zeta\frac{\partial}{\partial b}(\tilde{y}^T M(b)\tilde{y})$$

$$=[0.001944,\ 0.05678,\ -0.02076] \tag{48}$$

In case the generalized global formulation is employed, the same eigenvalue as in the reduced formulation is computed. The $M_g(b)$-normalized eigenvector is

$$y_g=[-0.3496,\ 0.08451,\ 0,\ 0,\ 0.2601,\ -0.45051]^T \tag{49}$$

The eigenvalue design sensitivity formula of Eq. (25), with the variational formulation, gives

$$\frac{d\zeta}{db}=\frac{\partial}{\partial b}(\tilde{y}_g^T K_g(b)\tilde{y}_g)-\zeta\frac{\partial}{\partial b}(\tilde{y}_g^T M_g(b)\tilde{y}_g)$$

$$=[0.001944,\ 0.05678,\ -0.02076] \tag{50}$$

which is the same as in Eq. (48).

Since there is no evidence of designs leading to repeated eigenvalues in this example, repeated eigenvalue sensitivity formulas are not written. For examples that exhibit repeated eigenvalues, the reader is referred to Ref. 3.

Acknowledgment

Research leading to this paper was supported by National Science Foundation Grant CEE80-05677.

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